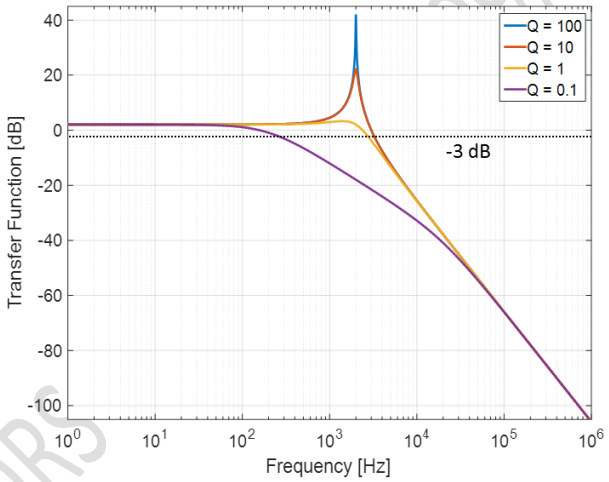
Question n. 1

"The choice of the optimum quality factor for a MEMS accelerometer is strictly related to its final application".

Explain in deep details if you agree with the above statement or not. Help yourself by discussing the system-level relationships between the Q factor and other relevant parameters for an accelerometer.

The transfer function between applied force (or acceleration) and displacement (or capacitive variation, apart from constant terms) in a MEMS accelerometer is described by a spring-mass-damper 2nd order equation, as shown in the figure.

$$\left|\frac{X(j\omega)}{F_{ext}(j\omega)}\right| = \left|\frac{1/m}{\left({\omega_0}^2 - \omega^2 + j\frac{\omega\omega_0}{Q}\right)}\right| = \frac{1/m}{\sqrt{({\omega_0}^2 - \omega^2)^2 + \left(\frac{\omega\omega_0}{Q}\right)^2}}$$



Depending on the value of the damping coefficient *b*, and thus of the quality factor *Q*, the system can show an underdamped response (Q>1) or an overdamped response (Q<1).

The bandwidth of a MEMS accelerometer is thus the first parameter related to the quality factor. In case of overdamped devices, the -3dB bandwidth can be calculated in the usual way, so it will fall approximately at the frequency of the 1st pole. A too low quality factor will cause a reduction in the bandwidth, as in the case of *Q* = 0.1 in the figure.

On the other side, for underdamped systems, the -3dB definition of the bandwidth somewhat loses meaning, as the transfer function may exceed the +3dB value by a large amount at resonance. This will cause long ringdown times with time constant τ in case of accidental shocks or other events which stimulate the device with a white spectrum (thus exciting also the resonance frequency).

$$\tau = \frac{Q}{\pi f_0}$$

Therefore, a too large quality factor is also detrimental for the device bandwidth.

In principle, without any other knowledge about the dependence of other parameters on the *Q* factor, one would thus choose an optimum value of Q in the order of 0.5 to unity (slightly above or below is still well acceptable).

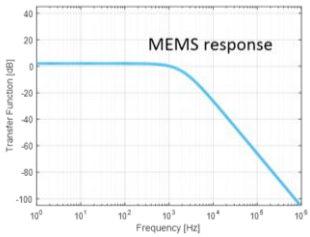
However, there is at least one further parameter that is not independent on *Q*, which is the device noise. To be more precise, noise is physically related to the damping coefficient *b*, but we know that also the *Q* factor, for a given mass and resonance frequency, is related with an inverse proportionality to *b*. The expression of the NEAD can be indeed written as:

$$NEAD = \sqrt{\frac{4\,k_B T\,b}{m^2}} = \sqrt{\frac{4\,k_B T\,\omega_0}{m\,Q}}$$

So, a relatively large value of the damping coefficient, compatible with Qs in the order of unity, will bring a large native Brownian noise.

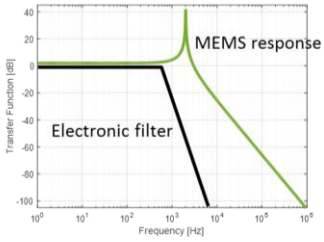
At this point, we can give a motivated answer to our original question and statement.

In case my application is compatible with noise values obtainable at *Q* around unity, e.g. in low-grade consumer applications, where electronic noise, due to power consumption constraints, usually dominates over Brownian noise, effectively the optimum *Q* factor of an accelerometer will be in the order of 0.5 to 1.



On the contrary, in case my application requires very good noise performance and has no significant power (i.e. electronic noise) constraints (e.g. geophones for geological analysis, medical applications, …) there is no alternative way other than lowering the damping coefficient to lower Brownian noise. Likely, the Q factor exceeds unity by a large amount.

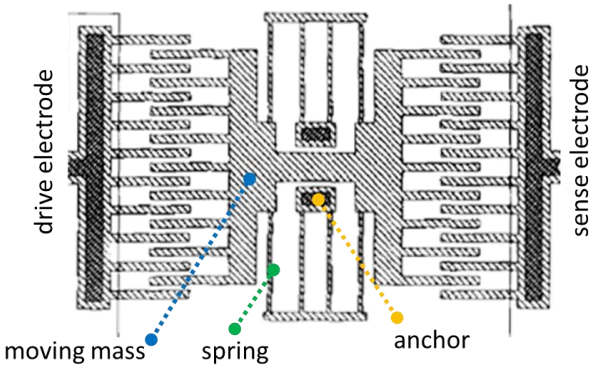
The solution to avoid long ringdown traces, appearing at the device output in case of shock events, is to place an electronic filter well before the resonance frequency, so to filter out the amplification of the peak. In other words, we will be forced to reduce our maximum sensing bandwidth to reach the desired noise density.



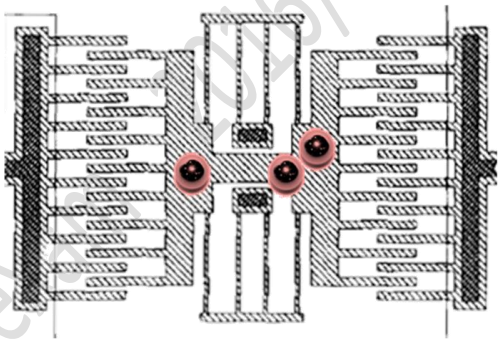
Question n. 2

A Tang resonator is used as a resonant sensor of molecules. The working principle is based on this simple concept: when a molecule is captured by the sensor, as in the figure below on the right, the value of the moving mass changes and determines a change of its dynamic response. The resonator surface is functionalized to capture only a specific type of molecule, of known mass m_{mol}, listed in the Table together with the native Tang resonance frequency. The resonator is embedded in a simple oscillating circuit, with no amplitude control. The whole system is kept within a controlled temperature package, as indicated.

Tang resonator with no captured particles



Tang resonator with 3 captured particles



1) find an analytical expression for the relative resonance frequency change $\left(\frac{df_0}{f_0}\right)$ of the oscillator per relative resonant mass change $\left(\frac{dm}{m}\right)$;

2) find an analytical expression for the relative voltage oscillation amplitude $\left(\frac{dV_0}{V_0}\right)$ change per relative mass change $\left(\frac{dm}{m}\right)$, considering the voltage V_0 at the output of the first stage, assuming both a charge amplifier and a trans-resistance amplifier;

Parameter	Symbol	Value
Mass of a single molecule	m_{mol}	$6 \cdot 10^{-15}$ kg
Operating temperature range	T_{min}-T_{max}	20°C – 20.6°C
Resonance frequency at 20.3°C	f_0	198 kHz
Damping coefficient	b	$5 \cdot 10^{-8}$ kg/s

3) based on the expressions found above and on the required operating conditions, which method (among those analyzed in points 1 and 2) would you choose for molecule detection?

4) for the chosen method, calculate the required resonator mass to discriminate a single molecule, and the corresponding stiffness and Q factor. For this calculation, assume that the circuit can measure the relative amplitude or frequency ($\frac{dV_0}{V_0}$ or $\frac{df_0}{f_0}$) changes with a resolution of $7.5 \cdot 10^{-6}$.

1.

The relationship between the suspended mass and the resonance frequency in an oscillating system is given by:

$$f_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

One can simply differentiate it, to find the answer to the first question, and evaluate that the relative resonance frequency change is half of the relative mass change:

$$\frac{df_0}{f_0} = -\frac{1}{2}\frac{dm}{m}$$

2.

In an oscillator without amplitude gain control, the force F applied to the driving port has always the same amplitude, regardless the amplitude of the oscillatory displacement x. This can be thus found through the quality factor Q and the stiffness k as:

$$x = \frac{F}{k}Q = \frac{F}{k}\frac{k}{\omega_0 b} = \frac{F}{\omega_0 b}$$

Now, the motional current at the sensing port of the resonator can be conveniently written as the product of the resonator velocity v and the transduction factor h. In terms of amplitude of the harmonic components, the velocity itself can be expressed as the product of the displacement and the angular resonance frequency:

$$i_m = \eta\, v = \eta\, x\, \omega_0$$

At this point we begin distinguishing between a TIA front-end, for which this current is passed through a feedback resistor to the first stage output, and a CA, where the current passes through a capacitive feedback impedance:

$$V_{out,TIA} = i_m R_F = \eta\, x\, \omega_0\, R_F = \eta\, \omega_0\, R_F\, \frac{F}{\omega_0 b} = \eta\, R_F\, \frac{F}{b}$$

$$V_{out,CA} = i_m \frac{1}{sC_F} = \eta\, x\, \omega_0\, \frac{1}{sC_F} = \eta\, \frac{F}{\omega_0 b}\, \omega_0\, \frac{1}{\omega_0 C_F} = \eta\, \frac{F}{b}\, \frac{1}{\omega_0 C_F}$$

We note therefore that in the case of the TIA stage, there is no dependence of the output voltage amplitude on the mass change. On the contrary, for the CA stage we have the following dependence:

$$\frac{dV_{out,CA}}{V_{out,CA}} = -\frac{d\omega_0}{\omega_0} = -\frac{df_0}{f_0} = \frac{1}{2}\frac{dm}{m}$$

3.

Excluding the TIA solution which leads to no result, the solution of measuring the relative frequency change and the solution of measuring the relative voltage change at the CA output are (in modulus) equivalent.

To choose among the two techniques, one can look at the required operating conditions: the temperature of the system is controlled but with a residual variation of about ±0.3 K.

As we know that the damping coefficient *b* (appearing in the formula of the CA output voltage) changes with temperature much more than the frequency itself, we may prefer to choose the solution based on the resonance frequency readout.

This is true but note that, in general, temperature changes are very slow event (with time constants usually in the seconds to minutes range), while the capture of a molecule will be a quasi-instantaneous event. So, the application of a high-pass filter to our readout will in principle compensate effects of temperature also in the voltage amplitude measurement.

4.

As we have chosen the first method, we can now easily calculate the maximum mass value that copes with the resolution of our readout:

$$7.5 \cdot 10^{-6} < \frac{1}{2}\frac{dm}{m} \quad \rightarrow \quad m < \frac{1}{2}\ \frac{m_{mol}}{7.5 \cdot 10^{-6}} = 0.4\ nkg$$

The corresponding values of stiffness and Q factor are easily found through the known formulas:

$$k = \omega_0^2\, m = 619\frac{N}{m}$$

$$Q = \frac{k}{\omega_0\, b} = 9950$$

Question n. 3

You are a test engineer of an imaging sensor company. You are asked to characterize noise parameters for a specific pixel of a CMOS sensor prototype, which operates under a 3 V supply voltage. To do this, you take a picture with an "*intermediate*" integration time, and you measure the analog output voltage variation of the selected pixel. You obtain, after offset compensation, an output voltage variation ΔV_{out} = 700.6 mV, with a noise contribution σ_{out} = 9.3 mV_{rms}. You are asked to:

- clarify the meaning of the word "*intermediate*" in the text above, and determine the number of electrons collected by the pixel;
- determine the value of the integration capacitance (assuming that the output voltage variation is equal to the voltage drop at the pixel integration node);
- draw the quoted photon transfer curve (PTC) in terms of number of electrons, assuming that the dark current and the FPN are negligible.

Physical Constants
q = 1.6 10^{-19} C
k_b = 1.38 10^{-23} J/K
T = 300 K (if not specified)
ε_0 = 8.85 10^{-12} F/m
$\varepsilon_{r,Si}$ = 11.7

From the Photon Transfer theory, we know that, if the input of a system is shot noise limted, i.e. if $\boldsymbol{SIGNAL = N}$ and $\boldsymbol{NOISE = \sigma_{noise} = \sqrt{N}}$, one can infer the transfer function from input to output as

$$\boldsymbol{K = \frac{\sigma_{out}^2}{\Delta V_{out}}}$$

In our case, K is the transfer function from the number of electrons and the output of the pixel, and

$$\boldsymbol{K = 123.45\ \mu V/electron}$$

The number of collected electrons can be thus evaluated as

$$\boldsymbol{N_{el} = \frac{\Delta V_{out}}{K} = 5675\ electrons}$$

As

$$\Delta V_{diode} = \Delta V_{out} = \frac{\Delta Q}{C_{int}} = \frac{qN_{el}}{C_{int}}$$

the integration capacitance can be estimated as

$$C_{int} = \frac{qN_{el}}{\Delta V_{diode}} = 1.29\ \text{fF}$$

The maximum number of electrons that can be collected can be easily calculated as

$$N_{max} = \frac{V_{DD}C_{int}}{q} = 24300\ \text{electrons}$$

The resulting photon transfer curve is therefore: